

BANACH SPACES WHICH EMBED INTO THEIR DUAL

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Abstract. We provide a nice characterization of the classical Riesz-Frechet representation theorem: if a Banach space isometrically embeds into its dual space, under two natural assumptions, then it is a Hilbert space and the embedding is nothing but the canonical one, which is surjective. On the other hand, we also prove that, by requiring surjectivity a priori, one can considerably weak the second supplementary assumption.

Anyway, it remains to prove that our assumptions are the best possible. It will be shown that the isometric hypothesis, in some sense, cannot be removed. In this context, we also provide a suitable example of compact convex Hausdorff space without a Borel measure with full support.

Finally, an example by Sztencel and Zaremba will be recalled to underline that it is impossible to completely remove the second supplementary property . Anyway, we do not know if also the first one is essential or not.

In this paper $(X, || \cdot ||)$ will denote a complex Banach space, the real case being just the same.

We start recalling the well known Birkhoff-James' definition of orthogonality in a Banach space (cfr. [2] and [5]).

Definition 1. $x \in X$ is said to be orthogonal to $y \in X$ if for each $\lambda \in \mathbb{C}$ one has

$$||x|| \leq ||x + \lambda y||$$

It is not difficult to verify that in the Hilbert space case this definition reduces to the usual one (given by the inner product). In the general context, where an inner product may not exist, it describes the following simple geometric property: x is orthogonal to y if and only if each triangle constructed on x and along y has the third side longer than the first one.

We point out that this is not the only definition of orthogonality available in a Banach

space, but surely it is the oldest and the most intuitive (for other useful notions of orthogonality, see [1], [10], [3] and [6]). Anyway, the main properties of Birkhoff-James' orthogonality are well known; in [3], for instance, Diminnie has shown that this orthogonality relation is forced to be reflexive if and only if X is a Hilbert space \mathcal{H} .

Before giving our first result, let us observe that the Riesz-Frechet isomorphism

$$\mathcal{H} \ni x \rightarrow f_x \in \mathcal{H}^* \quad \text{where } f_x(y) \doteq (x, y)^1 \quad \forall y \in \mathcal{H}$$

trivially verifies the property $x \in \text{Ker}(f_x)^\perp$, which can be required a priori in the context of a generic Banach space by using the Birkhoff-James orthogonality. In this way we can obtain the following

Theorem 2. *Let $(X, \|\cdot\|)$ be a complex normed space and $\Phi : X \rightarrow X^*$ an isometry such that*

1. $\langle \Phi(x), y \rangle = \overline{\langle \Phi(y), x \rangle} \quad \forall x, y \in X$
2. $x \in \text{Ker}(\Phi(x))^\perp \quad \forall x \in X \quad \text{in the sense of Birkhoff-James}$

Then X is a pre-Hilbert space with respect the inner product given by $(x, y) \doteq \langle \Phi(x), y \rangle$ and $(x, x) = \|x\|^2$ for all $x \in X$.

Proof. Clearly $(x, y) \doteq \langle \Phi(x), y \rangle$ defines a sesquilinear form on X , by virtue of the property 1. We will prove that this form is also positive definite. Let $x \in X$ be such that $(x, x) = 0$, then $x \in \text{Ker}(f_x)$ so 2) applies with $n = -x$: $\|x\| \leq 0$, i.e. $x = 0$. Now we observe that the real function $g : X \ni x \rightarrow \langle \Phi(x), x \rangle \in \mathbb{R}$ is continuous by triangle inequality, $X \setminus \{0\}$ is connected (unless $\dim X = 1$ and X real) and thus $\Phi(X \setminus \{0\})$ is an interval I not containing 0, whence $I \subseteq (-\infty, 0)$ or $I \subseteq (0, \infty)$. This shows that it is not restrictive to assume that $\langle \Phi(x), x \rangle > 0$ for each $x \neq 0$ (otherwise take $-\langle \Phi(x), x \rangle$). It remains to prove that $\langle \Phi(x), x \rangle \doteq (x, x) = \|x\|^2$. Clearly $\langle \Phi(x), x \rangle \leq \|\Phi(x)\| \|x\| = \|x\|^2$. Conversely, let $p(x)$ such that $\langle \Phi(x), x \rangle = p(x) \|x\|$. We have to prove that $p(x) \geq \|x\|$. Let $n \in \text{Ker}(\Phi(x))$ and $\lambda \in \mathbb{C}$, using 2) we have

$$|\langle \Phi(x), \lambda x + n \rangle| = |\lambda| \langle \Phi(x), x \rangle = |\lambda| p(x) \|x\| = p(x) \|\lambda x\| \leq p(x) \|\lambda x + n\|$$

¹According to the notations commonly used in Physics, we consider an inner product as a linear function in the second variable and a conjugate-linear in the first one.

Now we remember that when n runs $Ker(\Phi(x))$ and $\lambda \in \mathbb{C}$, $\lambda x + n$ describes the whole X , since $Ker(\Phi(x))$ has codimension 1 and does not contain x , so $\|x\| = \|\Phi(x)\| \leq p(x)$. \square

Remark 3. Conditions 1)-2) cannot be entirely removed; indeed it is not difficult to find a Banach space X isometrically isomorphic to its dual space X^* , without being a Hilbert space. To this aim it is enough to take $X = Y \oplus Y^*$, where Y is a reflexive non-Hilbert space; clearly we have $X \cong X^*$ under the isometric isomorphism $\Phi : X \rightarrow X^*$ given by $Y \oplus Y^* \ni (x, \varphi) \xrightarrow{\Phi} (\varphi, j(x)) \in Y^* \oplus Y^{**}$, $j : Y \rightarrow Y^{**}$ being the canonical injection of Y into its bidual space Y^{**} .

Remark 4. An interesting aspect of this proof is that it does not assume the surjectivity of the embedding $X \xrightarrow{\Phi} X^*$; it is instead a consequence of our milder hypothesis, at least when X is norm-complete. However, by assuming surjectivity, we are able to relax condition 2), as th.6 shows.

Remark 5. Another interesting aspect of this proof is the connectedness argument, which allows us to assume $\langle \Phi(x), x \rangle \geq 0$ without any loss of generality. It also allows us to generalize a result by Drivaliaris and Yannakakis (see [4], prop. 3.1), which follows by theorems 6 and 7 (indeed, if Φ is an isomorphism we do not need the reflexivity of X and the closedness of $Ran\Phi$ to prove th.7).

Theorem 6. *Let $\Phi : X \rightarrow X^*$ be an isometric isomorphism that satisfies the following*

1. $\langle \Phi(x), y \rangle = \overline{\langle \Phi(y), x \rangle}$
2. $x \in Ker(\Phi(x)) \Rightarrow x = 0$

Then X is a Hilbert space with respect to the inner product given by $(x, y) \doteq \langle \Phi(x), y \rangle$ and $(x, x) = \|x\|^2$.

Proof. The same argument of the previous proof shows that (\cdot, \cdot) is positive definite. Setting $|x| = (x, x)^{1/2}$, it only remains to prove that $|x| = \|x\|$, for each $x \in X$. Clearly $|x|^2 \leq \|\Phi(x)\| \|x\| = \|x\|^2$. Conversely, by the Hahn-Banach theorem, there exists $\varphi \in X^*$, with $\|\varphi\| = 1$, such that $\|x\| = \langle \varphi, x \rangle$. By the surjectivity of the embedding we have $\varphi = \Phi(y)$, for a unique $y \in X$ with $\|y\| = \|\Phi(y)\| = \|\varphi\| = 1$. So

$$\|x\| = \langle \Phi(y), x \rangle = (y, x) \leq |y| |x| \leq \|y\| \cdot |x| = |x|$$

in which the first inequality is nothing but the Cauchy-Schwarz inequality applied to (\cdot, \cdot) . \square

Now we want to end this part with a more refined result. Indeed, just assuming that $\text{Ran}\Phi$ is closed, under assumption of reflexivity, we get the same conclusion up to norm-equivalence. More precisely

Theorem 7. *Let $\Phi : X \rightarrow X^*$ be a bounded map from the reflexive Banach space X into its dual with closed range and such that*

1. $\langle \Phi(x), y \rangle = \overline{\langle \Phi(y), x \rangle}$
2. $x \in \text{Ker}(\Phi(x)) \Rightarrow x = 0$

Then the norm of X is equivalent to the Hilbert norm given by $|x| = \langle \Phi(x), x \rangle^{\frac{1}{2}}$.

Proof. We start observing that Φ is injective by 2, so it is a (conjugate-linear) isomorphism between X and $\text{Ran}\Phi$. Then, by the Banach inverse operator theorem we get $\|\Phi(x)\| \geq \delta\|x\|$ for each $x \in X$ (for some $\delta > 0$). As in the previous proofs we set $(x, y) = \langle \Phi(x), y \rangle$ and we easily get that it is a positive-definite sesquilinear form. Now

$$|x|^2 = \langle \Phi(x), x \rangle \leq \|\Phi(x)\| \|x\| \leq \|\Phi\| \|x\|^2$$

To prove the reverse inequality, we previously need to show the surjectivity of Φ . This is a straightforward consequence of the reflexivity of X : $\text{Ran}\Phi$ is a dense and closed subspace of X^* because $\text{Ran}\Phi^\perp$ (polar space of $\text{Ran}\Phi$) is the null space, as one can easily check. Now, let $x \in X$ and $\varphi \in X^*$, with $\|\varphi\| = 1$, such that $\|x\| = \langle \varphi, x \rangle$. As Φ is surjective, we have $\varphi = \Phi(y)$, for a unique $y \in X$. So, by virtue of the Cauchy-Schwarz inequality for the positive definite form (\cdot, \cdot)

$$\|x\| = \langle \Phi(y), x \rangle = (y, x) \leq |y| |x| \leq \|\Phi\|^{1/2} \|y\| |x| \leq \delta^{-1} \|\Phi\|^{1/2} |x|$$

This ends the proof. \square

The assumption of reflexivity for X can seem *ad hoc*; actually it is a quite natural request, as the following partial converse of the previous theorem shows

Theorem 8. *If X is a real Banach space and $\Phi : X \rightarrow X^*$ is an isometric isomorphism such that $\langle \Phi(x), y \rangle = \langle \Phi(y), x \rangle$ for all $x, y \in X$, then X is reflexive.*

Proof. It is a straightforward application of *James'* characterization of reflexivity. Let $\varphi \in X^*$ with $\|\varphi\| = 1$; we will prove that φ attains its norm on X_1 , the unit ball of X . Since Φ is a surjection, there is a (unique) $x \in X$ such that $\varphi = \Phi(x)$. We have

$$\|\varphi\| = \|\Phi(x)\| = \|x\|$$

By the Hahn-Banach theorem, there is $\eta \in X_1^*$ such that $\|x\| = \langle \eta, x \rangle$. Now let $y \in X$ such that $\eta = \Phi(y)$. We have $\|y\| = \|\eta\| = 1$ and $\|x\| = \langle \Phi(y), x \rangle = \langle \Phi(x), y \rangle = \langle \varphi, y \rangle$, hence $\|\varphi\| = \langle \varphi, y \rangle$. The last equality concludes the proof as $y \in X_1$ \square

Remark 9. The previous theorem has been already proved by Lin (see [8]) in the separable case, by using the *Eberlein-Šmulian* theorem.

Let us conclude with a few comments on the hypothesis. Certainly one cannot completely remove our assumptions on Φ , even if th.7 suggests an interesting observation. Indeed, we used the closedness of $\text{Ran}\Phi$ and the reflexivity of X only to prove that $C\|x\| \leq |x|$. Thus we have the following

Corollary 10. *Let X be a normed space and $\Phi : X \rightarrow X^*$ be a bounded map, verifying the following*

1. $\langle \Phi(x), y \rangle = \overline{\langle \Phi(y), x \rangle} \quad \forall x, y \in X$
2. $x \in \text{Ker}(\Phi(x)) \Rightarrow x = 0$

Then $(x, y) = \langle \Phi(x), y \rangle$ defines a pre-Hilbertian structure on X and the topology induced by (\cdot, \cdot) is weaker than norm-topology.

Instead, the following corollary generalizes a theorem by Lin ([8], th. 3).

Corollary 11. *Let X be a normed space and $\Phi : X \rightarrow X^*$ a bounded map such that*

$$\langle \Phi(x), x \rangle \geq m\|x\|^2, \quad \forall x \in X \quad (m \text{ a positive constant})$$

then X is isomorphic to a pre-Hilbert space.

Proof. We observe that $\tilde{\Phi}$ given by $\langle \tilde{\Phi}(x), y \rangle = \frac{1}{2}[\langle \Phi(x), y \rangle + \overline{\langle \Phi(x), y \rangle}]$ verifies the hypothesis of the previous corollary, so it applies to $\tilde{\Phi}$. \square

Let us denote \tilde{X} for the completion of X with respect to the inner product $\langle \Phi(x), y \rangle$. It is nice to calculate \tilde{X} in the following simple cases:

1. We consider the contraction of l^1 into its dual l^∞ given by the inclusion. It is easy to check the $\tilde{l}^1 = l^2$.
2. Let $L^1(B(\mathcal{H}))$ and $L^2(B(\mathcal{H}))$ be respectively the trace class and the Hilbert-Schmidt operator on a Hilbert space \mathcal{H} . $L^1(B(\mathcal{H}))$ is canonically embedded into its dual $B(\mathcal{H})$ through the conjugate-linear map $T \rightarrow tr(T^* \cdot)$ and $L^1(\widetilde{B(\mathcal{H})}) = L^2(B(\mathcal{H}))$.

Actually the (injective) contraction of a Banach space into a Hilbert space is nothing special at least in the separable case. Indeed the classical Banach-Mazur representation theorem gives a linear isometric embedding for every separable Banach space into $C[0, 1]$, which is obviously contracted into $L^2[0, 1]$ (with respect the Lebesgue measure). Maybe one can hope that the contraction of cor.10 is the most economic (in the sense that any other inner product weaker than the norm is also weaker than the previous one). On the other hand, what happens in the non-separable case is not clear: one can apply the Banach-Mazur theorem again to get an isometry from X onto a closed subspace of $C(X_1^*)$, X_1^* being the unit ball of the dual space of X with the $\sigma(X^*, X)$ topology. A way to obtain a canonical embedding now is given by taking a Borel positive measure μ , whose support is the whole X_1^* , and considering the natural embedding $C(X_1^*) \subset L^2(X_1^*, \mu)$. Here is a subtle problem: does any compact Hausdorff space K admit a (finite) Borel measure full supported? The answer is surely positive if K is metrizable (second countable), since the corresponding abelian C^* -algebra $C(K)$ is separable and thus it has a faithful state, corresponding (via the Riesz-Markov theorem) to such a measure. However, in the general case, the answer is negative even for very nice spaces, as the following counterexample shows:

Example 12. (Compact convex Hausdorff space which does not admit a Borel measure with full support) Let \mathcal{H} be a non-separable Hilbert space, $\{e_a, a \in A\}$ an orthonormal basis of \mathcal{H} and $K = \{x \in \mathcal{H} : \|x\| \leq 1\}$ with the weak topology. We set $U_a = \{x \in X : |(x, e_a)|^2 > \frac{1}{2}\}$. This is a non countable family of non empty ($e_a \in U_a$)

disjoint (by Parseval equality) open (since the real functions $x \rightarrow |(x, e_a)|^2$ are continuous with respect to the weak topology) sets. Thus, if μ is a Borel measure on K , $a \in A$ such that $\mu(U_a) = 0$ must exist (by σ -additivity) and thus $\text{supp}(\mu) \subseteq U_a^c$.

There is another pathological compact space that should be probably mentioned. We call “natural” a measure μ on K satisfying the following property: if $Y, Z \subseteq K$ are homeomorphic, then $\mu(Y)$ and $\mu(Z)$ are proportional. This definition guarantees that the measure is compatible with some dimension-concepts: if two subspaces are homeomorphic and one of those is μ -null, the other one must be the same.

Example 13. (Compact connected Hausdorff space which does not admit non-trivial finite and natural Borel measures) Let μ be a finite natural Borel measure on $K = [0, 1]^{\mathbb{R}}$, we will prove that μ is trivial, i.e. $\mu(K) = 0$. Let $\{R_x\}_{x \in \mathbb{R}}$ be a partition of \mathbb{R} such that $|R_x| = |\mathbb{R}|$ for each $x \in \mathbb{R}$ (see rem.14). Let \mathfrak{S} be the set of choice functions on the family $\{R_x\}$; $f, g \in \mathfrak{S}$ are called distinct if $f(x) \neq g(x), \forall x \in \mathbb{R}$. We choose a non-countable family f_α of pairwise distinct choice functions (see rem.14 again) and we consider the following non-countable family of subsets of K : $K_\alpha = [0, 1]^{\{f_\alpha(x), x \in \mathbb{R}\}}$. Since those are measurable (see rem.14) and pairwise disjoint (because f_α are distinct), then at least one of those is μ -null. But each one of those sets is isomorphic to K , and thus $\mu(K) = 0$.

Remark 14. In this remark we give some details on the previous example.

1. Firstly we prove that if X has the same cardinality of \mathbb{R} , then there exists a partition $\{R_x\}_{x \in \mathbb{R}}$ of X such that $|R_x| = |\mathbb{R}|$, for each $x \in \mathbb{R}$. Indeed, up to bijection, one can suppose $X = [0, 1] \times \mathbb{R}$ and take $R_x = [0, 1] \times \{x\}$.
2. Secondly we have to prove that there exists a non-countable family of pairwise distinct choice functions. This fact is not obvious and we will prove it in the next proposition.
3. Thirdly we prove that each K_α is a Borel subset of K . First of all we have to specify what we mean by “ K_α into K ”. We identify K_α with $[0, 1]^{\{f_\alpha(x), x \in \mathbb{R}\}} \times \{\frac{1}{2}\}^{\mathbb{R} \setminus \{f_\alpha(x), x \in \mathbb{R}\}}$. In this way we can represent K_α as the intersection of the following countable family of Borel subsets of K : $K_\alpha^n = [0, 1]^{\{f_\alpha(x), x \in R_x\}} \times [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]^{\mathbb{R} \setminus \{f_\alpha(x), x \in \mathbb{R}\}}$.

Proposition 15. *Let $\{X_a\}_{a \in A}$ be a family of sets and \mathfrak{S} the set of distinct choice function on this family. Then*

$$|\mathfrak{S}| = \inf\{|X_a|, a \in A\}$$

Proof. Certainly $|\mathfrak{S}| \leq \inf\{|K_a|, a \in A\}$. To prove the reverse inequality we assume that $|\mathfrak{S}| < \inf\{|K_a|, a \in A\}$, let $Y_a = K_a \setminus \{f(a), f \in \mathfrak{S}\}$. Each of those is non empty and thus we can construct a choice function distinct from those belonging to \mathfrak{S} . \square

Now we go back to our initial problem of establishing if it is possible to relax conditions 1) and 2). The following example ([11], see also [7] and [9]) proves that condition 2) cannot be removed: \mathbb{R}^3 with the norm $\|(x, y, z)\|_{SZ} = \max\{|y| + |z|, |x| + \frac{1}{2}|z|\}$ is isometric to its dual under the map $F(x, y, z) = (z, y, x)$, but one can prove that $(\mathbb{R}^3, \|\cdot\|_{SZ})$ is not a Hilbert space. Note that $F(1, 0, 0) \cdot (1, 0, 0) = 0$ and thus 2) is not verified. On the other hand we think that it should be impossible to completely remove condition 1), but maybe it could be assumed without loss of generality: it could exists $T \in B(X)$, isometric and onto, such that $\Phi \circ T$ verifies 1). In this direction, a partial result is due to Drivaliaris and Yannakakis (see [4], prop. 4.4).

We want to conclude this paper with the following comments:

1. In [10] and [1] other notions of orthogonality in normed spaces can be found. One can ask which among these allow us to get th.2 again.
2. We compare a paper by Partington with the present one. In [9] Partington shows that the existence of an orthogonality relation in a Banach space X satisfying some properties (which in general are not fulfilled by the Birkhoff-James orthogonality) implies that X is a Hilbert space and so there exists an isometric conjugate-linear isomorphism $\Phi : X \rightarrow X^*$, namely the Riesz-Frechet isomorphism. On the contrary, we solve the problem from the opposite viewpoint: given such an isomorphism $\Phi : X \rightarrow X^*$, which properties should be assumed for Φ in order to ensure that X is a Hilbert space and Φ is nothing but the Riesz-Frechet representation? The answers we gave in this paper are the following : it is enough that

- Φ is hermitian, isometric, surjective and $\langle \Phi(x), x \rangle = 0$ implies $x = 0$

- Φ is hermitian, isometric and $x \in \text{Ker}(\Phi(x))^\perp$ in the sense of Birkhoff-James implies $x = 0$.

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